Université de Douala
Ecole Nationale Supérieure Polytechnique
Année academique 2023-2024
Sciences de l'ingénieur& Ingénieur
Fonctions Spéciales et Polynômes orthogonaux
.
Dr. P. Njionou Sadjang

Fiche d'exercices numéro 3.

Exercice 1

Soit (p_n) une famille de polynomes orthogonaux non moniques. On pose $p_n(x) = k_n x^n + \ldots$. La famille (p_n) vérifie la relation de récurrence à trois termes

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_n(x), \quad n = 1, 2, \ldots.$$

Déterminer a_n et c_n en fonction de k_n uniquement. Que vaut b_n ?

Exercice 2

We consider the orthogonal polynomial sequence $(P_n)_{n\geq 0}$ defined for all integer n by the following Rodrigues formula :

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

- Find the weight function $\rho(x)$ associated to the family $(P_n)_{n\geq 0}$.
- Find the second-order differential equation satisfied by P_n.
- Let S be a polynomial. We assume that for all integer n,

$$\int_{-1}^{1} S(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\frac{d^n S(x)}{dx^n}\times(x^2-1)^n dx.$$

  Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n dx & \text{if } n = m. \end{cases}$$

- We assume that

$$\int_{-1}^{1}(1-x^2)^n dx = \frac{(n!)^2}{(2n+1)!}.$$

  Show that

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}.$$

Exercice 3

We consider the sequence (U_n) defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

- Compute $U_0(x)$, $U_1(x)$, $U_2(x)$.
- Prove that for all integer n, we have

$$U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x).$$

- Deduce that U_n is a polynomial of degree exactly n.
- Evaluate

$$\int_0^{\pi} \sin(n\theta)\sin(m\theta)d\theta, \quad m,n \in \mathbb{N}.$$

- Prove that $(U_n)_{n\geq 0}$ is an orthogonal polynomial sequence on $(-1,1)$ with respect to a weight function ρ which is to be determined.
- Give the Rodrigues representation of $(U_n)_{n\geq 0}$.
- Give a second-order differential equation satisfied by U_n.
- Give the Pearson type differential equation satisfied by $\rho(x)$.

Exercice 4

We consider the two variable function $\phi(x,y) = \dfrac{1}{\sqrt{1-2xy+x^2}}$. We assume that on the set $\mathcal{E} = \{(x,y) \in \mathbb{R}^2,\ \ 2|x||y| + |x|^2 < 1\}$ ϕ can be developed as

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n(y)x^n, \tag{1}$$

where the A_ns are of class C^∞ and that the partial derivative of ϕ of any order, with respect to both variables x and y can be computed by taking the derivative term by term of the right-hand side of (1).

- Compute $A_n(0)$ and $A_n'(0)$ for each $n \in \mathbb{N}$.
- Compute $x\dfrac{\partial\phi}{\partial x}(x,y) + (x-y)\dfrac{\partial\phi}{\partial y}(x,y)$.
  Deduce that $yA_0'(y) = 0$ and $\forall n \geq 1$, $yA_n'(y) - A_{n-1}'(y) = nA_n(y)$.
- Compute $(1-2xy+x^2)\dfrac{\partial\phi}{\partial x}(x,y) + (x-y)\phi(x,y)$.
  Deduce that $A_1(y) - yA_0(y) = 0$ and for each $n \geq 2$

$$nA_n(y) - (2n-1)yA_{n-1}(y) + (n-1)A_{n-2}(y) = 0.$$

- - Prove that $A_n'(y) - yA_{n-1}'(y) = nA_{n-1}(y)$.
  - Prove that $(1-y^2)A_n''(y) - 2yA_n'(y) + n(n+1)A_n(y) = 0$.
- To which family belong the polynomials (A_n)?

Exercice 5

The Laguerre polynomials $L_n^\alpha(x)$ is defined by the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}). \tag{2}$$

- Prove the orthogonality relation

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{mn}, \quad \alpha > -1. \tag{3}$$

- Prove that the generating function for $L_n^{(\alpha)}(x)$ is given by

$$F(x,r) := \frac{1}{(1-r)^{\alpha+1}}\exp\left(\frac{-xr}{1-r}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)r^n. \tag{4}$$

- Using the generating function, prove for $n = 1, 2, 4, \ldots$ that

$$(n+1)L_{n+1}^{(\alpha)}(x) + (x-\alpha-2n-1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0. \tag{5}$$

- Prove for $n \geq 1$ that

$$x\frac{dL_n^{(\alpha)}}{dx} = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{6}$$

- Use the previous (and only the previous results) to prove that

$$x(L_n^{(\alpha)})''(x) + (\alpha+1-x)(L_n^{(\alpha)})'(x) + n(L_n^{(\alpha)})(x) = 0, \quad n \geq 0. \tag{7}$$

- Prove that the Hermite polynomials and the Laguerre polynomials are linked by

$$H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{-1/2}(x^2) \tag{8}$$
$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{1/2}(x^2). \tag{9}$$

- Prove the following elementary formulas :

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \tag{10}$$
$$\frac{d}{dx}\left[x^\alpha L_n^{(\alpha)}(x)\right] = (n+\alpha)x^{\alpha-1}L_n^{(\alpha-1)}(x), \tag{11}$$
$$\frac{d}{dx}\left[e^{-1}L_n^{(\alpha)}(x)\right] = -e^{-x}L_{n+1}^{(\alpha)}(x), \tag{12}$$
$$\frac{d}{dx}\left[x^\alpha e^{-x}L_n^{(\alpha)}(x)\right] = (n+1)x^{\alpha-1}e^{-x}L_{n-1}^{(\alpha-1)}(x). \tag{13}$$

- Find the coefficients $I_m(n)$ in the expansion

$$x^n = \sum_{m=1}^{n} I_m(n)L_m^{(\alpha)}(x). \tag{14}$$

Exercice 6

On définit le n-ième polynôme de Hermite par

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right).$$

On va definir pour deux polynômes P et Q,

$$\langle P|Q\rangle = \int_{-\infty}^{\infty} P(t)Q(t)e^{-\frac{t^2}{2}}dt.$$

- Montrer que $\langle .|.\rangle$ est bien un produit scalaire sur $\mathbb{R}[X]$.
- Soient $m, n \in \mathbb{N}$. Montrer que $n \neq m \implies \langle H_n, H_m\rangle = 0$.
- Soit $n \in \mathbb{N}$, calculer $\langle H_n|H_m\rangle$.
- Montrer qu'on a la relation de récurrence d'ordre 2 suivante :

$$\forall x \in \mathbb{R}, \quad H_{n+1} = xH_n(x) - nH_{n-1}(x).$$

- Montrer que L_n vérifie l'équation différentielle suivante :

$$\forall x \in \mathbb{R}, \quad H_n''(x) - xH_n'(x) + nH_n(x) = 0.$$

Exercice 7

Soit $n \in \mathbb{N}$.

- Montrer qu'il existe un unique polynôme $P \in \mathbb{C}[X]$ tel que $P(\cos\theta) = \cos n\theta)$ pour tout θ réel. On le note T_n.
- Trouver une relation entre T_{n-1}, T_n et T_{n+1}.
- Donner une équation différentielle vérifiée par T_n.
- Déterminer les racines de T_n.
- Calculer $\int_0^{\pi} \cos(n\theta)\cos(m\theta)d\theta$ pour tout $m, n \in \mathbb{N}$.
- Montrer que la famille (T_n) est orthogonale sur $(-1, 1)$ par rapport à une fonction poids ρ que l'on précisera.
- Donner l'équation différentielle de Pearson satisfaite par ρ.
- Quelle est la formule de Rodrigues pour les T_n ?

Exercice 8

Let $f(x)$ be a function. One defines the operator Δ (forward difference by

$$\Delta f(x) = f(x+1) - f(x). \tag{15}$$

and the operator ∇ (backward difference) by

$$\nabla f(x) = f(x) - f(x-1). \tag{16}$$

The Pocchammer symbol or shifted factorial is defined by

$$(\alpha)_0 := 1 \text{ and } \quad (\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1), \quad a \neq 0, \quad n = 1,2,3,\dots \tag{17}$$

The falling factorial is denoted by

$$a^{\underline{0}} := 1 \quad \text{and} \quad a^{\underline{n}} = a(a-1)(a-2)\dots(a-n+1), \quad n = 1,2,3,\dots.$$

The Gauss hypergeometric function is defined by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n. \tag{18}$$

The Charlier polynomials $C_n(x;a)$ are defined by the generating function

$$G(x,t) = e^t \left(1 - \frac{t}{a}\right)^x := \sum_{n=0}^{\infty} C_n(x;a) \frac{t^n}{n!}. \tag{19}$$

- Find a link between $(a)_n$ and $a^{\underline{n}}$.
- Prove that

$$C_n(a;a) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| \, -\frac{1}{a}\right). \tag{20}$$

- Prove the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x;a) C_n(x;a) = a^{-n} e^a n! \delta_{mn}, \quad a > 0. \tag{21}$$

- Prove the three-terms recurrence relation

$$a C_{n+1}(x;a) = (n + a - x) C_n(x;a) - n C_{n-1}(x;a). \tag{22}$$

- Prove that

$$\Delta C_n(x;a) = -\frac{n}{a} C_{n-1}(x;a). \tag{23}$$

- Prove that

$$\nabla \left[\frac{a^x}{x!} C_n(x;a)\right] = \frac{a^x}{x!} C_{n+1}(x;a). \tag{24}$$

- Prove that

$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^k C_k(x;a). \tag{25}$$

- Find the coefficients $D_k(n)$ in the expansion

$$C_n(x;b) = \sum_{k=0}^{n} D_k(n) C_k(x;a). \tag{26}$$